

THE NUMERICAL EQUIVALENCE RELATION FOR HEIGHT FUNCTIONS AND AMPLENESS AND NEFNESS CRITERIA FOR DIVISORS

CHONG GYU LEE

ABSTRACT. In this paper, we study properties of Weil height functions associated with numerically trivial divisors. It helps us to define the fractional limit of h_E with respect to h_D on U , with D ample:

$$\text{Flim}_D(E, U) := \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)}.$$

The value of $\text{Flim}_D(E, U)$ contains numerical information about a divisor E , enough to determine whether E is ample, numerically effective or pseudo-effective.

1. INTRODUCTION

Let V be a projective variety and let E be a divisor on V . In this paper, we determine geometric properties of a divisor E using a Weil height function h_E associated with E .

Theorem A. *Let V be a projective variety defined over a number field k , let U be a subset of V and let E, D be divisors on V with D ample. We define the fractional limit of h_E with respect to h_D on U to be*

$$\text{Flim}_D(E, U) := \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)}.$$

The fractional limit satisfies the following properties:

- (1) E is ample if and only if $\text{Flim}_D(E, V) > 0$.
- (2) E is numerically effective if and only if $\text{Flim}_D(E, V) \geq 0$.
- (3) E is effective only if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.
- (4) E is pseudo-effective only if $\text{Flim}_D(E, U) \geq 0$ for some U which is an infinite intersection of open subsets of V .
- (5) E is pseudo-effective if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.
- (6) Suppose that any pseudo-effective divisor on V is linearly equivalent to a sum of an effective divisor and a numerically effective divisor. Then, E is pseudo-effective if only if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.

Date: November 29, 2011.

1991 Mathematics Subject Classification. Primary: 11G50, 14G40 Secondary: 37P30, 14C20, 14C22.

Key words and phrases. Weil height machine, height, numerical equivalence, ample, numerically effective, effective.

Note that a pseudo-effective divisor is a limit of a sequence of effective divisors. (See Definition 2.13.)

A fundamental tenet of diophantine geometry is that the geometric properties of an algebraic variety should determine its basic arithmetic properties. We can see a good example in Weil's work: Weil [15] introduced one of the main ingredients of diophantine geometry, the Weil height machine. It is a group homomorphism between the Picard group of a projective variety V defined over a number field k and the group of equivalence classes of real valued functions on $V(\overline{k})$,

$$h : \text{Pic}(V) \rightarrow \{f : V(\overline{k}) \rightarrow \mathbb{R}\} / \sim$$

sending E to a Weil height function h_E associated with a divisor E on V . (See Definition 2.7 for the equivalence relation of real valued functions.) It is known that

$$\begin{aligned} E \text{ is torsion} &\implies |h_E| \text{ is bounded,} \\ E \text{ is ample} &\implies h_E \text{ is bounded below on } V, \\ E \text{ is effective} &\implies h_E \text{ is bounded below on some dense open set on } U \subset V. \end{aligned}$$

Conversely, it is also possible to extract geometric information about E from arithmetic properties of h_E . For example, let C be a projective curve and let E_C, D_C be divisors on C with $\deg(D_C) \geq 1$. Then, it is known that

$$\lim_{h_{C,D_C}(P) \rightarrow \infty} \frac{h_{C,E_C}(P)}{h_{C,D_C}(P)} = \frac{\deg E_C}{\deg D_C}.$$

(See [14, Theorem B.3.5].) Thus, E_C is ample (respectively numerically effective) if the above limit is positive (respectively nonnegative).

Theorem A is a generalization of this result to higher dimensions. Because of the relation between the intersection theory and Weil height functions, $\text{Flim}_D(E, U)$ looks like a lower bound for $D \cdot C$ for all effective curves C on V which properly intersect with $V \setminus U$. Indeed, let V be a projective variety defined over a number field k , let E be a divisor on V and let W be a subvariety of V of dimension m . We can consider $h_E|_W$ in following ways: let $\iota : W \rightarrow V$ be a closed embedding. Then, we get

$$h_E(\iota(P)) = h_{\iota^*E}(P) + O(1).$$

By intersection theory, ι^*D is a divisor on W , isomorphic to a $(m-1)$ -cycle $W \cdot E$ on V . In particular, if we choose an irreducible curve C on V such that $C \setminus U$ is a finite set of points, then we get

$$\frac{E \cdot C}{D \cdot C} = \lim_{\substack{P \in C \\ h_{V,D}(P) \rightarrow \infty}} \frac{h_{V,E}(P)|_C}{h_{V,D}(P)|_C} \geq \liminf_{\substack{P \in U \\ h_{V,D}(P) \rightarrow \infty}} \frac{h_{V,E}(P)}{h_{V,D}(P)}.$$

Therefore, the fractional limit provides numerical information enough to determine whether E is ample, numerically effective or pseudo-effective.

If E_1 and E_2 generate equivalent Weil height functions, their fractional limits are the same. Also, by the algebraic equivalence property of the Weil height machine (Theorem 2.6 (6)), $\text{Flim}_D(E, U)$

depends only on the algebraic equivalence class of E . So, we need to study the kernel of the Weil height machine and properties of Weil height functions associated with algebraically trivial divisors. In Section 3, we show that the kernel of the Weil height machine is the torsion subgroup of $\text{Pic}(V)$ using the Albanese variety (see §2.4 for details on Albanese varieties). It helps us to study properties of Weil height functions associated with algebraically or numerically trivial divisors. In Section 4, we define the algebraic and the numerical equivalence of Weil height functions to show that the fractional limit actually depends on the numerical equivalence class of E . In Section 5, we show some key lemmas which complete the proof of Theorem A.

Acknowledgements. I would like to thank Joseph H. Silverman for his overall advice. I also thanks Dan Abramovich for his helpful instruction in algebraic geometry, especially for background on movable curves.

2. PRELIMINARIES

In this section, we introduce several algebro-geometric definitions and theorems required later. Generally, we will adopt notations from [14] unless stated otherwise.

2.1. Divisors and Weil height functions on projective varieties.

Definition 2.1. Let D_1, D_2 be divisors on a projective variety V . We say D_1 is algebraically equivalent to D_2 if there exist a connected algebraic set T , two points $t_1, t_2 \in T$ and a divisor \mathfrak{D} on $V \times T$ such that

$$D_i = \mathfrak{D}|_{V \times \{t_i\}}$$

for $i = 1, 2$. And we denote it by $D_1 \equiv D_2$. Also, we say D_1 is numerically equivalent to D_2 if

$$D_1 \cdot C = D_2 \cdot C$$

for all irreducible curves C on V and write $D_1 \equiv_n D_2$. For notational convenience, we say D is algebraically trivial (respectively numerically trivial) if $D \equiv 0$ (respectively $D \equiv_n 0$).

Definition 2.2. Let V be a projective variety. We define the Néron-Severi Group of V to be the group of algebraic equivalence classes of divisors on V :

$$\text{NS}(V) = \text{Div}(V) / \equiv .$$

Also, we define the numerical Néron-Severi Group of V to be the group of numerical equivalence classes of divisors on V :

$$N^1(V) = \text{Div}(V) / \equiv_n .$$

Remark 2.3. $\text{NS}(V), N^1(V)$ are abelian groups of finite rank. More precisely, $N^1(V)$ is the free part of $\text{NS}(V)$.

Definition 2.4. Let V be a smooth projective variety and let D be a divisor on V . We say that D is very ample if there is a closed embedding $\iota : V \rightarrow \mathbb{P}^N$ such that

$$D = \iota^* H \quad \text{where } H \text{ is a hyperplane of } \mathbb{P}^N.$$

We say that D is ample if mD is very ample for some positive integer m .

Definition 2.5. Let h be the logarithmic absolute height function on $\mathbb{P}^n(\overline{\mathbb{Q}})$, let V be a smooth projective variety defined over a number field k and let D be a very ample divisor on V . Then, we define a Weil height function on V associated with D to be

$$h_{V,D}(P) := h(\iota_D(P))$$

where $\iota_D : V \rightarrow \mathbb{P}^n$ is the closed embedding defining given very ample divisor D .

Theorem 2.6 (The Weil height machine). Let V be a smooth projective variety defined over a number field k . Then there exists a map between $\text{Div}(V)$, the group of divisors on V and the group of real valued functions on V

$$h_V : \text{Div}(V) \longrightarrow \{\text{functions } : V(\overline{k}) \rightarrow \mathbb{R}\}$$

with the following properties:

- (1) (Normalization) Let $H \subset \mathbb{P}^n$ be a hyperplane and let h be the absolute logarithmic height on $\mathbb{P}^n(\overline{k})$. Then,

$$h_{\mathbb{P}^n,H}(P) = h(P) + O(1) \quad \text{for all } P \in \mathbb{P}^n(\overline{k}).$$

- (2) (Functoriality) Let $\phi : V \rightarrow W$ be a morphism and let $D \in \text{Div}(W)$. Then,

$$h_{V,\phi^*D}(P) = h_{W,D}(\phi(P)) + O(1) \quad \text{for all } P \in V(\overline{k}).$$

- (3) (Additivity) Let $D, E \in \text{Div}(V)$. Then,

$$h_{V,D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1) \quad \text{for all } P \in V(\overline{k}).$$

- (4) (Linear Equivalence) Let $D, E \in \text{Div}(V)$ with D linearly equivalent to E . Then,

$$h_{V,D}(P) = h_{V,E}(P) + O(1) \quad \text{for all } P \in V(\overline{k}).$$

- (5) (Positivity) Let $D \in \text{Div}(V)$ be an effective divisor and let B be the base locus of the linear system $|D|$. Then,

$$h_{V,D}(P) \geq O(1) \quad \text{for all } P \in (V \setminus B)(\overline{k}).$$

- (6) (Algebraic Equivalence) Let $D, E \in \text{Div}(V)$ with D ample and E algebraically trivial. Then,

$$\lim_{\substack{P \in V(\overline{k}) \\ h_{V,D}(P) \rightarrow \infty}} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0.$$

- (7) (Finiteness) *Let $D \in \text{Div}(V)$ be an ample divisor. Then for every finite extension k'/k and every constant C , the set*

$$\{P \in V(k') \mid h_{V,D}(P) \leq C\}$$

is finite.

- (8) (Uniqueness) *The height functions $h_{V,D}$ are uniquely determined up to $O(1)$ by (1) Normalization, (2) Functoriality and (3) Additivity.*

Proof. See [14, Theorem B.3.2]. □

Definition 2.7. *Let V be a projective variety defined over a number field k and let $h_1, h_2 : V \rightarrow \mathbb{R}$ be real valued functions on V . We say h_1 is equivalent to h_2 if*

$$h_1(P) = h_2(P) + O(1)$$

and write

$$h_1 \sim h_2.$$

We also define the group of equivalence classes of Weil height functions on V to be

$$\text{Ht}(V(\bar{k})) := \{h_{V,D} \mid D \in \text{Div}(V(\bar{k}))\} / \sim.$$

2.2. Ampleness, numerical effectiveness and pseudo-effectiveness. We refer [2] to the reader for details. We will assume that V is a projective variety and D, E are divisors on V for notational convenience in this subsection.

Theorem 2.8 (Nakai-Moishezon Criterion). *A divisor D is ample if and only if*

$$D^r \cdot Y > 0 \quad \text{where } r \text{ is the dimension of } Y$$

for all integral subvariety Y of V .

Theorem 2.9 (Kleiman's Criterion). *A divisor D is ample if and only if*

$$D \cdot C > 0$$

for all C in closure the cone of effective curves on V .

Proof. See [6] or [2, Theorem 1.27]. □

Lemma 2.10. *Let V be a projective variety and let D, E be ample divisors on V . Then, there is a positive real number $m > 0$ such that $mD - E$ is ample again.*

Proof. See [14, Theorem A.3.2.3]. □

Definition 2.11 (Numerical effectiveness). *A divisor D is called numerically effective if*

$$D^r \cdot Y \geq 0 \quad \text{where } r \text{ is the dimension of } Y$$

for all integral subvariety Y of V .

Theorem 2.12. *A divisor D is numerically effective if and only if*

$$D \cdot C \geq 0$$

for all irreducible curves C on V .

Proof. See [2, Theorem 1.26]. □

Definition 2.13. *We define the effective cone of V to be*

$$\text{Eff}(V) = \langle E \mid E \text{ is an effective divisor} \rangle \otimes \mathbb{R}.$$

And, we call a divisor E pseudo-effective if $E \in \overline{\text{Eff}(V)}$.

Definition 2.14. *A curve $C \subset V$ is called movable if there exists an irreducible algebraic family of curves*

$$\{C_s \mid s \in S\}$$

containing C as a reduced member and dominating V .

Theorem 2.15 (Boucksom-Demailly-Paun-Peternell). *Let V be a projective variety and let E be a divisor on V . Then, E is pseudo-effective if and only if*

$$E \cdot C \geq 0$$

for all irreducible movable curves C on V .

Proof. See [1, Section 2]. □

Corollary 2.16. *If E is numerically equivalent to a pseudo-effective divisor, then E is pseudo-effective.*

Proof. Suppose that $D \equiv_n E$ and D is pseudo-effective. Then, for any irreducible movable curve C ,

$$E \cdot C = D \cdot C \geq 0.$$

Therefore, because of Theorem 2.15, E is pseudo-effective. □

2.3. Abelian Varieties. We refer [7] to the reader for details on abelian varieties.

Definition 2.17. Let A be an abelian variety and let D be a divisor on A . We say that D is symmetric if $[-1]^*D$ is linearly equivalent to D .

Definition 2.18. Let A be an abelian variety defined over a number field k and let D be a symmetric ample divisor on A . We define the canonical height function on $A(\bar{k})$ associated with D to be

$$\hat{h}_{A,D}(a) := \lim_{m \rightarrow \infty} \frac{1}{2^m} h_{A,D}([2^m]a)$$

where $h_{A,D}$ is a Weil height function associated with D .

Proposition 2.19. Let A be an abelian variety defined over a number field k and let D be a symmetric ample divisor on A . Then,

$$\hat{h}_{A,D}(a) = h_{A,D}(a) + O(1).$$

Proof. See [14, Theorem B.5.6]. □

Lemma 2.20. Let A be an abelian variety defined over a number field k and let \hat{h}_D be a canonical height function on A associated with a symmetric ample divisor D . Then we have

$$\hat{h}_D(a) = 0 \quad \text{if and only if} \quad a \text{ is of finite order.}$$

Proof. See [14, Proposition B.4.2, Theorem B.5.1]. □

Lemma 2.21. Let A be an abelian variety defined over a number field k and let D be an ample divisor on A . Then, \hat{h}_D is a quadratic form and hence we have a corresponding bilinear form:

$$\langle \cdot, \cdot \rangle_D : A(\bar{k}) \times A(\bar{k}) \rightarrow \mathbb{R}, \quad \langle a, b \rangle_D = \frac{1}{2} \left(\hat{h}_{A,D}(a+b) - \hat{h}_{A,D}(a) - \hat{h}_{A,D}(b) \right).$$

Moreover,

$$\langle a, b \rangle_D = 0$$

for all $b \in A$ if and only if a is of finite order.

Proof. See [14, Proposition B.5.3] to check \hat{h}_D is a quadratic form.

Let a be of order m . Then, $m\langle a, b \rangle_D = \langle ma, b \rangle_D = \langle O, b \rangle_D = 0$. On the other hand, if a is not finite order, then

$$\langle a, a \rangle_D = \hat{h}_D(a) \neq 0.$$

□

Proposition 2.22. Let A be an abelian variety and let D be a divisor on A . Let $t_a : A \rightarrow A$ is a translation endomorphism by $a \in A$:

$$t_a(b) = a + b \quad \text{for all } b \in A.$$

Then, there is a group homomorphism

$$\Phi_D : A \rightarrow \text{Pic}(A), \quad a \mapsto t_a^* D - D$$

satisfying

- (1) The image of Φ_D lies in $\text{Pic}^0(A)$.
- (2) If the divisor D is ample, then Φ_D is surjective and has finite kernel.

Proof. See [14, Theorem A.7.3.1]. □

Proposition 2.23. *Let A, B be abelian varieties defined over a number field k and let $\phi : B \rightarrow A$ be a morphism. Then,*

$$\widehat{h}_{B, \phi^* D}(b) = \widehat{h}_{A, D}(\phi(b)) - \widehat{h}_{A, D}(\phi(O)).$$

Proof. See [14, Theorem B.5.6]. □

Lemma 2.24. *Let A be an abelian variety, let D be a divisor on A and let t_a be a translation map by a . Then, we have*

$$t_{a+b}^* D + D \sim t_a^* D + t_b^* D \quad \text{for all } a, b \in A.$$

Proof. [14, Theorem A.7.2.9] □

2.4. Albanese Varieties. We refer [7, II, §2] for details on Albanese varieties.

Definition 2.25. *Let V be a projective variety. Then, we define the Albanese variety of V to be*

$$\text{Alb}(V) := H^0(V, \Omega_V^1)^* / H^1(V, \mathbb{Z})$$

where $H^0(V, \Omega_V^1)^*$ is the dual of $H^0(V, \Omega_V^1)$ and Ω_V^1 is the set of 1-forms on V .

Remark 2.26. $\text{Alb}(V)$ is an initial object: if there is any morphism $\phi : V \rightarrow A$ where A is an abelian variety, then there is a morphism $\psi : \text{Alb}(V) \rightarrow A$ such that $\phi = \psi \circ \pi$ where $\pi : V \rightarrow \text{Alb}(V)$ is the universal map.

Proposition 2.27. *Let V be a projective variety, let $A = \text{Alb}(V)$ and let $\pi : V \rightarrow A$ be the universal map from V to its Albanese variety. Then, the pullback map $\pi^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(V)$ is an isomorphism.*

Proof. See [7, Section IV.4 and VI.1, Theorem 1]. □

Theorem 2.28. *Let V be a projective variety, let $A = \text{Alb}(V)$ and let $\pi : V \rightarrow A$ be the universal map from V to its Albanese variety. Then there exists a positive integer N such that*

$$F : V \times \cdots \times V \rightarrow A, \quad F(P_1, \dots, P_N) = \sum_{i=1}^N \pi(P_i)$$

is a surjective map.

Proof. See [7, Theorem 11]. □

3. THE KERNEL OF THE WEIL HEIGHT MACHINE

Serre [10] showed that the kernel of the Weil height machine is the torsion subgroup of the Picard Group. But, his proof is quite simple so that we can't get further information. In this section, we introduce another proof using Albanese varieties, which helps studying properties of Weil height functions associated with algebraically trivial or numerically trivial divisors.

Lemma 3.1. *Let V be a projective variety defined over a number field k and let E be a divisor on V . Suppose that E is algebraically trivial. Then $|h_E(P)|$ is bounded if and only if E is of finite order in $\text{Pic}(V)$.*

Proof. One direction is clear: if E is of order m , then

$$h_E(P) = \frac{1}{m} h_{mE}(P) = \frac{1}{m} h_O(P) + O(1) = O(1).$$

For the other direction, let $A = \text{Alb}(V)$. Then, we have the universal map $\pi : V \rightarrow A$ such that the group homomorphism $\pi^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(V)$ is an isomorphism by Proposition 2.27. Moreover, by definition of algebraically trivial divisors (Definition 2.1), an algebraically trivial divisor E is connected to the zero divisor. Thus, $E \in \text{Pic}^0(V)$ and hence there is a divisor $E_0 \in \text{Pic}^0(A)$ such that $E = \pi^* E_0$ by Proposition 2.27.

Choose a symmetric ample divisor E_S on A . Then, we have a translation map

$$\Phi_{E_S} : A \rightarrow \text{Pic}^0(A)$$

which is surjective (Proposition 2.22). Hence, there is a point $a_0 \in A$ such that $E_0 = \Phi_{E_S}(a_0) = t_{a_0}^* E_S - E_S$.

Furthermore, we get

$$\begin{aligned}
 h_{V,E}(P) &= h_{V,\pi^* E_0}(P) + O(1) \\
 &= h_{A,E_0}(a) + O(1) && (a = \pi(P)) \\
 &= \widehat{h}_{A,E_0}(a) + O(1) && (\because \text{Proposition 2.19}) \\
 \text{(A)} \quad &= \widehat{h}_{A,t_{a_0}^* E_S}(a) - \widehat{h}_{A,E_S}(a) + O(1) && (\because E_0 = t_{a_0}^* E_S - E_S) \\
 &= \widehat{h}_{A,E_S}(t_{a_0}(a)) - \widehat{h}_{A,E_S}(t_{a_0}(O)) - \widehat{h}_{A,E_S}(a) + O(1) && (\because \text{Lemma 2.20}) \\
 &= \widehat{h}_{A,E_S}(a + a_0) - \widehat{h}_{A,E_S}(a_0) - \widehat{h}_{A,E_S}(a) + O(1) \\
 &= 2\langle a, a_0 \rangle_{E_S} + O(1).
 \end{aligned}$$

Thus, the boundedness of $h_{V,E}$ guarantees that $\langle \pi(P), a_0 \rangle$ is bounded for all $P \in V$.

By Theorem 2.28, there exists a positive integer N such that

$$F : \prod_{i=1}^N V \rightarrow \text{Alb}(V), \quad (P_1, \dots, P_N) \mapsto \sum_{i=1}^N \pi(P_i)$$

is a surjective map. So, $\langle \cdot, a_0 \rangle_{E_S}$ is bounded on A :

$$|\langle a, a_0 \rangle_{E_S}| \leq \sum_{i=1}^N |\langle \pi(P_i), a_0 \rangle_{E_S}| \leq O(1) \quad \text{for all } a \in A.$$

Since it is a bilinear map, the boundedness means trivial: $\langle [m]a, a_0 \rangle_{E_S} = m \langle a, a_0 \rangle_{E_S}$ is bounded for all m and hence $\langle a, a_0 \rangle_{E_S} = 0$. So, a_0 is of finite order by Lemma 2.21.

Therefore, $E_0 = \Phi_{E_S}(a_0)$ is also of finite order and hence $E = \pi^*E_0$ is of finite order because Φ_{E_S}, π^* are group homomorphisms. \square

Corollary 3.2. *Let V be a projective variety defined over a number field k and let E be a divisor on V with E numerically trivial. Then, $|h_E(P)|$ is bounded if and only if E is of finite order in $\text{Pic}(V)$.*

Proof. One direction is clear: if E is of finite order then $|h_E(P)|$ is bounded.

Suppose that E is numerically trivial and $|h_E(P)|$ is bounded. Then, there is a positive integer m such that $mE \equiv 0$ (Remark 2.3). Because $|h_{mE}(P)|$ is still bounded, mE is of finite order and hence so is E . \square

Theorem 3.3. *The kernel of the Weil height machine*

$$h_V : \text{Pic}(V) \rightarrow \text{Ht}(V(\overline{k}))$$

is the torsion subgroup of $\text{Pic}(V)$

Proof. One direction is clear: a torsion element of $\text{Pic}(V)$ generates a bounded Weil height function.

Suppose that $|h_E(P)|$ is bounded. It is enough to show that $E \equiv_n 0$ because of Corollary 3.2. Take an ample divisor D on V and an irreducible curve C on V . By assumption, $|h_E(P)|$ is bounded on $C \subset V$ and hence

$$\frac{E \cdot C}{D \cdot C} = \frac{\deg \iota^*E}{\deg \iota^*D} = \lim_{h_{\iota^*D}(Q) \rightarrow \infty} \frac{h_{\iota^*E}(Q)}{h_{\iota^*D}(Q)} = \lim_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} = 0$$

where $\iota : C \rightarrow V$ is the closed embedding of C into V and Q is a point on C . By Kleiman's Criterion (Theorem 2.9), we get $D \cdot C > 0$ and hence $E \cdot C = 0$. Therefore, $E \equiv_n 0$. \square

Corollary 3.4.

$$h_{D_1} \sim h_{D_2} \quad \text{if and only if} \quad D_1 - D_2 \sim E \text{ for some } E \in (\text{Pic}(V))_{\text{tor}}.$$

Corollary 3.5. *Let V be a projective variety defined over a number field k and let E be a divisor on V . Suppose that E is algebraically trivial. Then, either*

- (1) $|h_E|$ is bounded, or
- (2) h_E is not bounded above nor below.

Proof. Suppose that $E \equiv 0$ and $|h_E|$ is not bounded. Then, by (A) in the proof of Theorem 3.1, we have

$$h_E(P) = 2\langle a, a_0 \rangle_{E_S} + O(1)$$

where $a = \pi(P)$ and $E = \pi^* t_{a_0}^* E_S$. Recall that a_0 is a point of infinite order if E is not torsion. By Theorem 2.28, there exists a positive integer N such that

$$F : \prod_{i=1}^N V \rightarrow \text{Alb}(V), \quad (P_1, \dots, P_N) \mapsto \sum_{i=1}^N \pi(P_i)$$

is generically surjective. So, there is a sequence of N -tuples of points $T_m = (P_{1,m}, \dots, P_{N,m})$ such that

$$F(T_m) = [m]a_0.$$

If $\langle \pi(P_{i,m}), a_0 \rangle_{E_S} < \frac{m}{N} \hat{h}_{E_S}(a_0)$ for all $m > 0$ and $i = 1, \dots, N$. Then we get

$$\begin{aligned} m \hat{h}_{E_S}(a_0) &= \sum_{i=1}^N \langle \pi(P_{i,m}), a_0 \rangle_{E_S} \\ &\leq \sum_{i=1}^N \langle \pi(P_{i,m}), a_0 \rangle_{E_S} \\ &< m \hat{h}_{E_S}(a_0) \end{aligned}$$

which is a contradiction. Hence, there is an i_m for each $m > 0$ satisfying

$$\langle \pi(P_{i_m, m}), a_0 \rangle_{E_S} \geq \frac{m}{N} \hat{h}_{E_S}(a_0).$$

Therefore, we get

$$\lim_{m \rightarrow \infty} h_E(P_{i_m, m}) \geq \lim_{m \rightarrow \infty} \frac{m}{N} \hat{H}_{E_S}(a_0) + O(1) \geq \infty.$$

Similarly, there is a j_m for each $(-m) < 0$ satisfying

$$\langle \pi(P_{j_m, -m}), a_0 \rangle_{E_S} \leq \frac{-m}{N} \hat{h}_{E_S}(a_0)$$

and hence we get

$$\lim_{m \rightarrow \infty} h_E(P_{j_m, -m}) \leq \lim_{m \rightarrow \infty} \frac{-m}{N} \hat{H}_{E_S}(a_0) + O(1) \leq -\infty.$$

□

4. THE NUMERICAL AND ALGEBRAIC EQUIVALENCES OF HEIGHT FUNCTIONS

In this section, we define algebraic and numerical equivalence relations of Weil height functions. Though the algebraic and the numerical equivalence relations of divisors are strictly different, but the algebraic and the numerical equivalence relations of Weil height functions are same. So, we can have another description of the numerical Néron-Severi group. Also, we can see that the numerical equivalence relation of Weil height functions shows numerical information of corresponding divisors.

Definition 4.1. Let V be a projective variety defined over a number field k , let D_1, D_2 be divisors on a projective variety V and let h_i be a Weil height function associated with D_i . We say h_1 is algebraically equivalent to h_2 if

$$D_1 - D_2 \equiv E \quad \text{for some torsion element } E \in \text{Pic}(V)$$

and write $h_1 \equiv h_2$. Also, we say h_1 is numerically equivalent to h_2 if $D_1 \equiv_n D_2$ and write $h_1 \equiv_n h_2$.

Proposition 4.2. Let h_1, h_2 be Weil height functions on V associated with D_1 and D_2 respectively. Then

- (1) $h_1 \equiv h_2$ if $h_1 \sim h_2$.
- (2) $h_1 \equiv_n h_2$ if $h_1 \sim h_2$.
- (3) $h_1 \equiv_n h_2$ if and only if $h_1 \equiv h_2$.
- (4) If $h_1 \equiv_n h_2$, then

$$\lim_{n \rightarrow \infty} \frac{h_1(P_n)}{h_D(P_n)} = \lim_{n \rightarrow \infty} \frac{h_2(P_n)}{h_D(P_n)}$$

for any sequence $\{P_n\} \subset V$ and any ample divisor D on V .

Proof. (1) If $h_1 \sim h_2$, then $D_1 - D_2 \sim E$ for some torsion element E in $\text{Pic}(V)$ so that $D_1 - D_2 \equiv E$.

- (2) Suppose $h_1 \sim h_2$. Then, $h_{D_1 - D_2}$ is a bounded height so that $D_1 - D_2$ is a torsion element of $\text{Pic}(V)$ of finite order m . Therefore, for any curve C on V ,

$$m(D_1 - D_2) \cdot C = O \cdot C = 0$$

and hence $D_1 \cdot C = D_2 \cdot C$.

- (3) One direction is clear: D is numerically equivalent to E if D is algebraically equivalent to E . Moreover, if E is of finite order, then E is numerically trivial.

For the other direction, consider the following diagram;

$$\begin{array}{ccc} \text{Pic}(V)/(\text{Pic}(V))_{\text{tor}} & \xrightarrow{\sim} & \text{Ht}(V(\overline{k})) \\ \pi_a \downarrow & & \pi_a \downarrow \\ \text{NS}(V)/(\text{NS}(V))_{\text{tor}} & \xrightarrow{\sim} & \text{Ht}(V(\overline{k}))/\equiv \\ \pi_n \downarrow & & \pi_n \downarrow \\ N^1(V)/(N^1(V))_{\text{tor}} & \xrightarrow{\sim} & \text{Ht}(V(\overline{k}))/\equiv_n \end{array}$$

We know that all horizontal group homomorphisms are isomorphisms. Furthermore,

$$\text{NS}(V)/(\text{NS}(V))_{\text{tor}} = N^1(V) \simeq N^1(V)/(N^1(V))_{\text{tor}}$$

so that corresponding height groups are also isomorphic:

$$\mathrm{Ht}(V(\overline{k}))/\equiv_n \simeq \mathrm{Ht}(V(\overline{k}))/\equiv.$$

- (4) Suppose that $h_1 \equiv_n h_2$ and D is an ample divisor. Then, there is a nonzero integer m satisfying $mE = mD_1 - mD_2 \equiv 0$. By the algebraic equivalence property of the Weil height machine (Theorem 2.6,(6)), we get

$$\lim_{h_D(P) \rightarrow \infty} \frac{h_{mE}(P)}{h_D(P)} = 0.$$

Therefore,

$$(B) \quad 0 = \lim_{h_D(P) \rightarrow \infty} \frac{1}{m} \frac{h_E(P)}{h_D(P)} = \lim_{h_D(P) \rightarrow \infty} \left(\frac{h_1(P)}{h_D(P)} - \frac{h_2(P)}{h_D(P)} \right)$$

and hence both $\lim_{n \rightarrow \infty} \frac{h_1(P_n)}{h_D(P_n)}$ and $\lim_{n \rightarrow \infty} \frac{h_2(P_n)}{h_D(P_n)}$ diverge or converge to the same number.

(Note that the limit on (B) is defined but $\lim_{h_D(P) \rightarrow \infty} \frac{h_1(P)}{h_D(P)}$ may not be defined so that we will treat limsup and liminf in Section 5.)

□

5. AMPLE DIVISORS AND THE FRACTIONAL LIMIT OF WEIL HEIGHTS

In this section, we will prove the main theorem. Proposition 4.2 (4) shows that

$$\liminf_{h_D(P) \rightarrow \infty} \frac{h_1(P)}{h_D(P)} = \liminf_{h_D(P) \rightarrow \infty} \frac{h_2(P)}{h_D(P)}, \quad \limsup_{h_D(P) \rightarrow \infty} \frac{h_1(P)}{h_D(P)} = \limsup_{h_D(P) \rightarrow \infty} \frac{h_2(P)}{h_D(P)}$$

if $h_1 \equiv_n h_2$. We will show that they are actually finite numbers having numerical information of corresponding divisors. We start by showing the choice of an ample divisor D does not affect to the desired result.

Lemma 5.1. *Let V be a projective variety defined over a number field k and let D_1, D_2 be ample divisors on V . Then, for any sequence $\{P_n\} \subset V(\overline{k})$,*

$$\lim_{n \rightarrow \infty} h_{D_1}(P_n) = \infty \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} h_{D_2}(P_n) = \infty.$$

Proof. Suppose that there is a sequence of points $\{P_n\}$ such that

$$\lim_{n \rightarrow \infty} h_{D_1}(P_n) = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} h_{D_2}(P_n) < M$$

for some positive number M . Find m such that $mD_2 - D_1$ is ample (Lemma 2.10) and get

$$\liminf_{h_{D_1}(P) \rightarrow \infty} \frac{h_{mD_2 - D_1}(P)}{h_{D_1}(P)} \leq \limsup_{n \rightarrow \infty} \frac{h_{mD_2 - D_1}(P_n)}{h_{D_1}(P_n)} = -1.$$

However, the Weil height function associated with an ample divisor $mD_2 - D_1$ is bounded below (Theorem 2.6 (7)) so that

$$\liminf_{n \rightarrow \infty} \frac{h_{mD_2 - D_1}(P_n)}{h_{D_1}(P_n)} \geq 0,$$

which is a contradiction. \square

Proposition 5.2. *Let V be a projective variety defined over a number field k and let D_1, D_2 be ample divisors on V . Then,*

$$0 < \liminf_{h_{D_2}(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_{D_2}(P)} \leq \limsup_{h_{D_2}(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_{D_2}(P)} < \infty.$$

Proof. Since D_1 and D_2 are ample, there is a constant $m_1 > 0$ such that $m_1 D_1 - D_2$ is ample again by Lemma 2.10. Weil height functions associated with ample divisors are bounded below (Theorem 2.6 (7)) and hence $h_{D_1}(P) > \frac{1}{m_1} (h_{D_2}(P) - O(1))$. Therefore,

$$(C) \quad \liminf_{h_{D_2}(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_{D_2}(P)} = \liminf_{h_{D_2}(P) \rightarrow \infty} \frac{1}{m_1} \frac{h_{D_2}(P) - O(1)}{h_{D_2}(P)} = \frac{1}{m_1} > 0.$$

Because of the Lemma 5.1, we can change the limit and hence we get

$$(D) \quad \limsup_{h_{D_2}(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_{D_2}(P)} = \frac{1}{\liminf_{h_{D_2}(P) \rightarrow \infty} \frac{h_{D_2}(P)}{h_{D_1}(P)}} = \frac{1}{\liminf_{h_{D_1}(P) \rightarrow \infty} \frac{h_{D_2}(P)}{h_{D_1}(P)}} < \infty.$$

Finally, combine (C) and (D) to get the desired result. \square

Corollary 5.3. *Let V be a projective variety defined over a number field k and let D, E be divisors on V with D ample. Then, we have*

$$\limsup_{h_D(P) \rightarrow \infty} \left| \frac{h_E(P)}{h_D(P)} \right| < \infty.$$

Proof. Suppose that $E \sim D_1 - D_2$ where D_i are ample divisors. Then, by Proposition 5.2, we get

$$\limsup_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} \leq \limsup_{h_D(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_D(P)} - \liminf_{h_D(P) \rightarrow \infty} \frac{h_{D_2}(P)}{h_D(P)} < \limsup_{h_D(P) \rightarrow \infty} \frac{h_{D_1}(P)}{h_D(P)} < \infty.$$

The other inequality is easily gained from the first inequality:

$$- \liminf_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} = \limsup_{h_D(P) \rightarrow \infty} \frac{h_{-E}(P)}{h_D(P)} < \infty.$$

\square

Proposition 5.4. *Let V be a projective variety defined over a number field k and let E be a divisor on V . Then, the followings are equivalent:*

$$(1) \quad \lim_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} = 0 \text{ for all ample divisor } D.$$

- (2) $\lim_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} = 0$ for some ample divisor D .
- (3) $E \equiv_n 0$.

Proof. (1) \Rightarrow (2) is clear and (3) \Rightarrow (1) is shown in Proposition 4.2 (5).

Show (2) \Rightarrow (1) first. Suppose that (2) is true for an ample divisor D_1 . Then, for any other ample divisor D_2 , we get

$$\limsup_{h_{D_2}(P) \rightarrow \infty} \left| \frac{h_{D_1}(P)}{h_{D_2}(P)} \right| = C < \infty$$

by Proposition 5.2. Furthermore, we can change the limit because of Lemma 5.1. Therefore, we get

$$\lim_{h_{D_2}(P) \rightarrow \infty} \left| \frac{h_E(P)}{h_{D_2}(P)} \right| \leq \limsup_{h_{D_2}(P) \rightarrow \infty} \left| \frac{h_{D_1}(P)}{h_{D_2}(P)} \right| \cdot \lim_{h_{D_1}(P) \rightarrow \infty} \left| \frac{h_E(P)}{h_{D_1}(P)} \right| = C \cdot 0$$

and hence (1) holds for D_2 .

Now we prove (1) \Rightarrow (3): suppose that E is not numerically trivial. Then, there is an irreducible curve C such that $E \cdot C \neq 0$. Pick an ample divisor D . Then, by Nakai-Moishezon Criterion (Theorem 2.8), we get $D \cdot C > 0$. Thus,

$$\lim_{\substack{P \in C \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C}{D \cdot C} \neq 0$$

so (1) fails. Hence, (1) holds only if (3) holds. \square

Now we can prove the main theorem of this paper.

Theorem A. *Let V be a projective variety defined over a number field k , let U be a subset of V and let E, D be divisors on V with D ample. We define the fractional limit of h_E with respect to h_D on U to be*

$$\text{Flim}_D(E, U) := \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)}.$$

The fractional limit satisfies the following properties:

- (1) E is ample if and only if $\text{Flim}_D(E, V) > 0$.
- (2) E is numerically effective if and only if $\text{Flim}_D(E, V) \geq 0$.
- (3) E is effective only if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.
- (4) E is pseudo-effective only if $\text{Flim}_D(E, U) \geq 0$ for some U which is an infinite intersection of open subsets of V .
- (5) E is pseudo-effective if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.
- (6) Suppose that any pseudo-effective divisor on V is linearly equivalent to a sum of an effective divisor and a numerically effective divisor. Then, E is pseudo-effective if and only if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.

Remark 5.5. *The condition on (6) is a necessary condition for various versions of Zariski decomposition. (For details of Zariski decomposition, see [1, 5, 8].) For example, a projective surface S allows Zariski decomposition so that a divisor E on S is pseudo-effective if and only if $\text{Flim}_D(E, U) \geq 0$ for some dense open set $U \subset V$.*

Proof. (1) The ‘only if’ part is proved by Proposition 5.2. So, it’s enough to show the ‘if’ part. Suppose that

$$\liminf_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} = \alpha > 0.$$

Then, for any effective cycle $C \in \text{Eff}(V)$, we may assume that

$$C = \sum_{j=1}^n \beta_j C_j$$

where β_j are positive real numbers and C_j are irreducible curves. Then, for any curve C_j ,

$$\alpha \leq \liminf_{\substack{h_D(P) \rightarrow \infty \\ P \in C_j}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C_j}{D \cdot C_j}$$

so that

$$E \cdot C = \sum_{j=1}^n \beta_j (E \cdot C_j) \geq \sum_{j=1}^n \beta_j (\alpha D \cdot C_j) \geq \alpha \sum_{j=1}^n \beta_j > 0.$$

Moreover, for any cycle $C' \in \overline{\text{Eff}}(V) \setminus \text{Eff}(V)$, there a sequence C_k of nonzero effective cycles converging to C' . Suppose that $C_k = \sum_{j=1}^{j_k} \beta_{j,k} C_{j,k}$ where $C_{j,k}$ are irreducible curves and $\beta_{j,k}$ are positive real numbers. Since $C_k \cdot E > 0$, we get $C' \cdot E \geq 0$. Moreover, if $C' \cdot E = 0$, then

$$0 = \lim_{k \rightarrow \infty} C_k \cdot E \geq \alpha \sum_{j=1}^{j_k} \beta_{j,k} \geq 0$$

so that $\lim_{k \rightarrow \infty} \sum_{j=1}^{j_k} \beta_{j,k} = 0$ and hence $C' = \lim C_k = 0$. Therefore, if C' is nonzero, then $C' \cdot E > 0$ and hence E is ample by Kleiman’s Criterion.

(2) Let E be a numerically effective divisor. Then, by Kleiman’s Criterion, $E_m = E + \frac{1}{m}D$ is ample for all $m > 0$ and all ample divisor D . Thus, by (1), we have

$$\liminf_{h_D(P) \rightarrow \infty} \frac{h_{E_m}(P)}{h_D(P)} > 0.$$

Therefore, $\text{Flim}_D \geq 0$ because the following inequality holds for all $m > 0$:

$$\liminf_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} \geq \liminf_{h_D(P) \rightarrow \infty} \frac{h_{E_m}(P)}{h_D(P)} - \frac{1}{m} \limsup_{h_D(P) \rightarrow \infty} \frac{h_D(P)}{h_D(P)} \geq -\frac{1}{m}.$$

Conversely, if E is not numerically effective, then there is a curve C such that $E \cdot C < 0$. Therefore,

$$\liminf_{\substack{P \in V \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \leq \liminf_{\substack{P \in C \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C}{D \cdot C} < 0.$$

(3) If E is effective, then h_E is bounded below on $V \setminus |E|$, where $|E|$ is the base locus of E .

Thus, we get

$$\liminf_{\substack{P \in V \setminus |E| \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \geq \liminf_{\substack{P \in V \setminus |E| \\ h_D(P) \rightarrow \infty}} \frac{C}{h_D(P)} = 0.$$

(4) Suppose that E is pseudo-effective. Then, there is an ample divisor D_0 such that $E + \epsilon D_0$ is effective for any $\epsilon > 0$. Construct a set \mathbf{B} , an infinite union of a closed subsets of V :

$$\mathbf{B}(E) = \bigcup_{n=1}^{\infty} \left| E + \frac{1}{n} D_0 \right|$$

where $B_n = \left| E + \frac{1}{n} D_0 \right|$ is the base locus of $E + \frac{1}{n} D_0$. Since a Weil height function associated with an effective divisor is bounded below outside of the base locus (Theorem 2.6 (5)), we get

$$\liminf_{\substack{P \in V \setminus B_n \\ h_{D_0}(P) \rightarrow \infty}} \frac{h_E(P)}{h_{D_0}(P)} \geq 0.$$

Therefore, the following inequality holds on a set $U = V \setminus \mathbf{B}(E)$ which is an infinite intersection of open subsets of V for all n :

$$\begin{aligned} \liminf_{\substack{P \in U \\ h_{D_0}(P) \rightarrow \infty}} \frac{h_E(P)}{h_{D_0}(P)} &= \liminf_{\substack{P \in U \\ h_{D_0}(P) \rightarrow \infty}} \left(\frac{h_{E+\frac{1}{n}D_0}(P)}{h_{D_0}(P)} - \frac{\frac{1}{n}h_{D_0}(P)}{h_{D_0}(P)} \right) \\ &\geq \liminf_{\substack{P \in U \\ h_{D_0}(P) \rightarrow \infty}} \frac{h_{E+\frac{1}{n}D_0}(P)}{h_{D_0}(P)} - \frac{1}{n} \\ &\geq -\frac{1}{n} \end{aligned}$$

and hence

$$\liminf_{\substack{P \in U \\ h_{D_0}(P) \rightarrow \infty}} \frac{h_E(P)}{h_{D_0}(P)} \geq 0.$$

Finally, for any ample divisor D , we get

$$\liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \geq \liminf_{\substack{P \in U \\ h_{D_0}(P) \rightarrow \infty}} \frac{h_E(P)}{h_{D_0}(P)} \times \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_{D_0}(P)}{h_D(P)} \geq 0.$$

- (5) Suppose that E is not pseudo-effective. Then, there is a movable curve C such that $E \cdot C < 0$ because of Theorem 2.15. Then, for any dense open set U of V , there is a curve C_t in the irreducible family $\{C_s \mid s \in S\}$ of C such that $C_t \cap U \neq \emptyset$. Moreover, $E \cdot C_t = E \cdot C < 0$ and hence

$$\liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \leq \liminf_{\substack{P \in U \cap C_t \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C_t}{R \cdot C_t} < 0.$$

Therefore, if there is a dense open set satisfying the following condition:

$$\liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \geq 0$$

then E is pseudo-effective.

- (6) Let E be a pseudo-effective divisor. If $E = E_P + E_N$ for some numerically effective divisor E_P and some effective divisor E_N , then, by (2) and (3), we get

$$\liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_E(P)}{h_D(P)} \geq \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_{E_P}(P)}{h_D(P)} + \liminf_{\substack{P \in U \\ h_D(P) \rightarrow \infty}} \frac{h_{E_N}(P)}{h_D(P)} \geq 0$$

where $U = V \setminus |E_P|$.

□

Example 5.6. Let V be a projective surface defined over a number field, let D be a ample divisor and let E be a pseudo-effective divisor. Then, V allows the Zariski decomposition and hence

E is pseudo-effective if and only if $\text{Flim}_D(E, U) \geq 0$ for some open set U .

Theorem 5.7. Let V, W be projective varieties defined over a number field k , let $\phi : W \rightarrow V$ be a dominant morphism and let D_W, D_V be ample divisors on W and V respectively. Let $\mu(\phi, D_W, D_V)$ be Silverman's height expansion coefficient of dominant map ϕ [12]:

$$\mu(\phi, D_W, D_V) := \sup_{U \subset W} \liminf_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{D_V}(\phi(P))}{h_{D_W}(P)}.$$

Then,

$$\mu(\phi, D_W, D_V) \leq \sup\{\alpha \mid \phi^* D_V - \alpha D_W \text{ is pseudo-effective.}\}.$$

Proof. Suppose $\mu(\phi, D_W, D_V) \geq \alpha$. Then, by definition of μ , there is a open set U such that

$$\liminf_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{D_V}(\phi(P))}{h_{D_W}(P)} \geq \alpha$$

Since

$$\lim_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{D_W}(P)}{h_{D_W}(P)} = 1,$$

$$\liminf_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{D_V}(\phi(P))}{h_{D_W}(P)} \geq \alpha \cdot \lim_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{D_W}(P)}{h_{D_W}(P)}$$

and hence

$$\liminf_{\substack{P \in U \\ h_{D_W}(P) \rightarrow \infty}} \frac{h_{\phi^* D_V - \alpha D_W}(P)}{h_{D_W}(P)} \geq 0.$$

Therefore, $\phi^* D_V - \alpha D_W$ is pseudo-effective. \square

Question 5.8. *We want to know Theorem A (6) holds in general: let $V(k)$ be a projective variety over a number field k and let D be an ample divisor on V . Suppose that we have a pseudo-effective divisor E on V . Then is there an open set U of V such that*

$$\liminf_{h_D(P) \rightarrow \infty, P \in U} \frac{h_E(P)}{h_D(P)} \geq 0?$$

REFERENCES

- [1] Boucksom, P., Demailly, J.-P., Paun, M., Peternell, T., *The pseudo-effective cone of compact Kähler manifold and varieties of negative Kodaira dimension*. preprint, arXiv:math/0405285, 2004.
- [2] Debarre, O., *Higher dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.
- [3] Demailly, J.-P. , *Multiplier ideal sheaves and analytic methods in algebraic geometry*, in: *School on Vanishing Theorems and Effective Results in Algebraic Geometry*. Trieste 2000, ICTP Lecture Notes Vol. 6 (Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001), pp. 1–148.
- [4] Fulton, W., *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. Springer-Verlag, Berlin, 1998.
- [5] Hartshorne, S., *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
- [6] Kleiman, S., *Toward a numerical theory of ampleness*. Ann. of Math. (2) 84 1966 293?344.
- [7] Lang, S., *Abelian Varieties*. Interscience Tracts in Pure and Applied Mathematics. No. 7 Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London 1959. (Reprint, Springer-Verlag, New York-Berlin, 1983.)
- [8] Lang, S., *Fundamentals of diophantine geometry*. Springer-Verlag, New York, 1983.
- [9] Lazarsfeld, R. , *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.
- [10] Serre, J. P., *Lectures on Mordell-Weil Theorem*. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt. With a foreword by Brown and Serre. Third edition. Aspects of Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [11] Shafarevich, I. *Basic algebraic geometry, 1. Varieties in projective space*. Second edition. Translated from the 1988 Russian edition and with notes by Miles Reid. Springer-Verlag, Berlin, 1994.
- [12] Silverman, J. H. *A height estimate for equidimensional dominant rational maps*, preprint, arXiv:0908.3835, 2009.
- [13] Silverman, J. H. *The arithmetic of dynamical systems*. Graduate Texts in Mathematics, 241. Springer, New York, 2007.

- [14] Silverman, J. H., Hindry, M. *Diophantine geometry, An introduction*. Graduate Texts in Mathematics, 201. Springer-Verlag, New York, 2000.
- [15] Weil, A., *Arithmetic on algebraic varieties*. Ann. of Math. (2) 53, (1951). 412–444.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, IL, 60607 US

E-mail address: `phiel@math.uic.edu`